

All finitely presentable groups from link complements and Kleinian groups

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Dedicated to Professor Shinji Fukuhara on his 65th birthday

Abstract

Klein defined geometry in terms of invariance under groups actions; here we give a discrete (partial) converse of this, interpreting all (finitely presentable) groups in terms of the geometry of hyperbolic 3-manifolds (whose fundamental groups are, appropriately, Kleinian groups). For G^* a Kleinian group of isometries of hyperbolic 3-space \mathbb{H}^3 , with $M_{G^*} \cong \mathbb{H}^3/G^*$ a non-compact N -cusped orientable 3-manifold of finite volume, let $\mathcal{P}_{G^*} \subset S_\infty^2 = \partial\mathbb{H}^3$ be its dense set of parabolic fixed points. Let $\bar{M}_{G^*} := \mathbb{H}^3 \cup \mathcal{P}_{G^*}/G^*$ be the 3-complex obtained by compactifying each cusp of M_{G^*} with an additional point. This is the 3-dimensional analogue of the standard compactification of cusps of hyperbolic Riemann surfaces. We prove that every finitely presentable group G is of the form $G = \pi_1(\bar{M}_{G^*})$, in infinitely many ways: thus every finitely presentable group arises as the fundamental group of an orientable 3-complex \bar{M} – denoted as a ‘link-singular’ 3-manifold – obtained from a hyperbolic link complement by coning each boundary torus of the link exterior to a distinct point.

We define the *closed-link-genus*, $clg(G)$, of any finitely presentable group G , which completely characterizes fundamental groups of closed orientable 3-manifolds: $clg(G) = 0$ if and only if G is the fundamental group of a closed orientable 3-manifold. Moreover $clg(G)$ gives an upper bound for the concept *genus*(G) of genus defined earlier by Aitchison and Reeves, and in turn is bounded by the minimal number of relations among all finite presentations of G .

Our results place some aspects of the study of finitely presentable groups more centrally within both classical and modern 3-manifold topology: accordingly, proofs given are expressed in these terms, although some can be seen naturally in 4-manifold topology.

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1 Introduction and preliminaries

For general background, we refer the reader to Lyndon and Schupp [22], Hempel [9], Rolfsen [28] and Thurston [31].

Presentations for groups, and their relationship with fundamental groups of topological complexes, has motivated much of the 20th century research into the classification of manifolds. The successful classification of the fundamental groups of surfaces naturally led to the desire to classify 3-manifold fundamental groups, and understand the extent to which uniqueness holds. In the case of simply-connected compact 3-manifolds, uniqueness is essentially equivalent to the Poincaré Conjecture, recently resolved by Perelman in his solution of Thurston’s Geometrization Conjecture for 3-manifolds. In principle, fundamental groups of compact 3-manifolds are now algorithmically classifiable (Bridson [5]). It has been known for decades that every finitely presentable group does arise (non-uniquely) as the fundamental group of some closed orientable 4-manifold. Not all finitely presentable groups arise as the fundamental groups of compact 3-manifolds – see for example, Kawauchi [16] or Shalen [30] – and moreover the isomorphism problem for finitely presentable groups is algorithmically unsolvable ([1, 27]): it is of interest to make more precise the nature of the distinction between compact 3-manifold-groups and arbitrary finitely presentable groups.

Quinn has shown that all finitely presentable groups arise as the fundamental groups of *non-orientable* 3-complexes, *allowing boundary*, but where all vertex links are either spheres or *projective planes*. This construction is elucidated in [12, 13]. In this paper we show they arise as the fundamental groups of *orientable* 3-complexes, with *empty* boundary, with all vertex links being either spheres or *tori*. As a consequence, the combinatorial group theory of finitely presentable groups can be set more centrally in the theory of orientable 3-manifolds, geometric structures and the theory of knots and links.

Every closed orientable surface can be triangulated; after deleting vertices, taking the universal cover topologically produces the ideal triangulation of the hyperbolic plane. Geometrically, we see every closed surface fundamental group arising from the compactification of a hyperbolic Riemann surface, with finitely-generated free fundamental group, by replacing cusp points. All closed (ie, compact with empty boundary) orientable topological 3-manifolds can be given an essentially unique piecewise-linear (‘PL’) structure – and hence can be viewed as a union of tetrahedra with all faces identified in pairs. Every PL-3-manifold admits an essentially unique smoothing as a differentiable manifold. Similar arguments show that every compact orientable 3-complex, obtained by arbitrary pairwise identification of faces of a disjoint union of tetrahedra, admits an essentially unique differentiable structure in the complement of its vertices. The following is well known (see for example [9, 31]):

Lemma 1. *For a compact orientable triangulated 3-complex M without boundary, obtained by pairwise face identifications of a finite number of tetrahedra, the following are equivalent:*

1. *The Euler characteristic $\chi(M)$ of M vanishes: $\chi(M) = 0$;*
2. *M is a 3-manifold;*
3. *all interior vertex links are 2-spheres.*

A simple weakening of the concept of (compact, orientable) 3-manifold is to allow finitely many vertex links which are the next simplest closed orientable surface, that is, a torus. (In dimension 2, a link can only be a circle.) Suppose M is such a 3-complex, and we delete an open cone neighbourhood of each singular point. The result is a 3-manifold M^* with non-empty boundary a finite union of tori, and every such manifold can be obtained from a closed orientable 3-manifold \bar{M} by deleting an open tubular neighbourhood of an embedded link $\mathcal{L} \subset \bar{M}$. Since coning a 2-sphere boundary component to a point does not change the fundamental group, in the following we generally consider compact orientable 3-manifolds with no 2-spheres in their boundary.

Remark. The Euler characteristic of a surface determines its possible constant-curvature geometries. Accordingly, spheres and tori play a major role in the structure of 3-manifolds: embedded spheres arise from π_2 , giving the prime decomposition of 3-manifolds; embedded tori yield the JSJ decomposition of 3-manifolds (also mirrored in group-theoretic constructions); and any aspherical, atoroidal closed orientable 3-manifold with infinite fundamental group admits a complete metric of negative curvature, according to Perelman-Thurston.

Lemma 2. *Suppose M is an arbitrary compact, orientable 3-manifold with $\partial M \neq \emptyset$, but containing no 2-sphere components. The following are equivalent:*

1. ∂M is a disjoint union of finitely many tori;
2. the Euler characteristic vanishes: $\chi(M) = 0$.

Let \mathcal{M}_χ^3 denote the set of compact, connected, orientable 3-manifolds M with Euler characteristic $\chi(M) = \chi$. Every finite-volume hyperbolic 3-manifold is uniquely the interior of some $M \in \mathcal{M}_0^3$.

Definition. A *link-singular* 3-manifold is any compact orientable 3-complex obtained from a compact orientable 3-manifold $M \in \mathcal{M}_0^3$ by attaching cones to some or all boundary tori: all interior vertex links are thus either spheres or tori.

For $M \in \mathcal{M}_0^3$, denote by $M_{C\partial}$ the orientable 3-complex obtained by coning each boundary torus to a distinct point (setting $M_{C\partial} \equiv M$ when $\partial M = \emptyset$). When $\mathcal{L} \subset M$ is a finite-component link in some closed orientable 3-manifold M , there is an open tubular neighbourhood $N\mathcal{L}$ giving the link exterior $M_\mathcal{L} := M - N\mathcal{L} \in \mathcal{M}_0^3$. We let $M_{C\mathcal{L}} := (M_\mathcal{L})_{C\partial}$ denote the link-singular 3-manifold obtained from $M_\mathcal{L}$ by attaching a cone to each boundary torus of the exterior.

Remark. Instead of coning boundary tori to points, we can attach a ‘solid torus’ or ‘donut’ – $S^1 \times D^2$ – to some or all boundary tori, to obtain a closed orientable 3-manifold. Any closed orientable 3-manifold M can be obtained from any given one M_0 by Dehn surgery on a link $\mathcal{L} \subset M_0$, which essentially means the deletion of solid tori neighbourhoods of all link components, and reattaching these by a possibly homotopically non-trivial homeomorphism of their boundary tori, specified by an integer assigned to each link component. This follows

from work 50 years ago by Rohlin, Lickorish and Wallace: Craggs and Kirby independently described the generators of the equivalence relation on the surgery instructions required to produce the same 3-manifold, notably in the case $M_0 = S^3$.

We give elementary knot-theoretic proofs of the following:

Theorem 1. *Let G denote an arbitrary finitely presentable group. Then $G \cong \pi_1(M_{C\mathcal{L}})$ for some link \mathcal{L} in a closed 3-manifold M , whose complement $M - \mathcal{L}$ admits a complete metric of constant curvature -1 and finite volume. Thus every finitely presentable group is the fundamental group of a link-singular 3-manifold, where the complement of all singular points admits a complete hyperbolic metric of finite volume.*

As a corollary, we obtain our main theorem, using the 3-dimensional analogue of the standard compactification of cusps of hyperbolic Riemann surfaces: For G^* any Kleinian group of isometries of hyperbolic 3-space \mathbb{H}^3 , with $M_{G^*} \cong \mathbb{H}^3/G^*$ a non-compact N -cusped orientable 3-manifold of finite volume, let $\mathcal{P}_{G^*} \subset S_\infty^2 = \partial\mathbb{H}^3$ be its dense set of parabolic fixed points. Let $\bar{M}_{G^*} := \mathbb{H}^3 \cup \mathcal{P}_{G^*}/G^*$ be the 3-complex obtained by compactifying each cusp of M_{G^*} with an additional point.

Theorem 2. *Every finitely presentable group G is of the form $G = \pi_1(\bar{M}_{G^*})$, for infinitely many Kleinian groups G^* .*

Remark. If G is the fundamental group of a closed orientable 3-manifold M , then G can be realized as the fundamental group of a closed link-singular 3-manifold with any even number of singular points by $\#$ -summing with some number of copies of ΣT , the suspension of a torus, by the Seifert-van Kampen Theorem, since $\pi_1(\Sigma T) = 1$.

Remark. That our main result might be true was motivated by the well-known result of Lickorish and Wallace that every closed orientable 3-manifold can be obtained by surgery on some link in the 3-sphere (see [31]), that the connect sum $M_n := \#_n S^1 \times S^2$ has fundamental group $\pi_1(M_n) \cong F_n$ (the free group of rank n), and that all finitely presentable groups arise as quotients of a free group.

We prove the theorem by constructing a very simple link \mathcal{L}_{P_G} in the 3-sphere S^3 from a given finite presentation P_G of a group G , analyzing the Wirtinger presentation of the fundamental group of its complement, and showing that killing off certain elements of the fundamental groups of peripheral tori yields the desired 3-complex with previously specified finite presentation of any arbitrary group. Similar links have arisen in related applications in the past, and accordingly we introduce a convenient more symmetric refinement of these constructions ([4, 14, 20, 21]).

2 Group presentations and link projections

When referring to a finite presentation we will consider both the generating set and the set of defining relations as ordered sets. Let $P_G = \langle X_1, X_2, \dots, X_n \mid R_1, R_2, \dots, R_k \rangle$ be a

finite (ordered) presentation for an arbitrary finitely presentable group G . Each relation R_j is thus a word $w_j(X_1, X_1^{-1}, \dots, X_n, X_n^{-1})$ in the monomials X_i, X_i^{-1} , and G is obtained from the free group F_n of rank n with presentation $P_n := \langle X_1, X_2, \dots, X_n \mid - \rangle$ by taking the quotient group F_n/N_R where $N_R = \langle\langle R_1, R_2, \dots, R_k \rangle\rangle$ is the normal closure of the set of relations.

We associate to P_G a link in S^3 as follows: Each relation R_j has a well defined length $l_j = |w_j|$, and we set $L = \sum_1^k l_k$, the total length of the presentation. It is well known that the complement of a disjoint union of t unknotted and unlinked circles has fundamental group F_t free of rank t . From such a link, we will change crossings to introduce Wirtinger relations and produce an appropriate Wirtinger presentation for the new link complement according to the conventions of Figure 1: labels correspond to generators of the fundamental group of the complement assigned to each overarc of a link diagram.

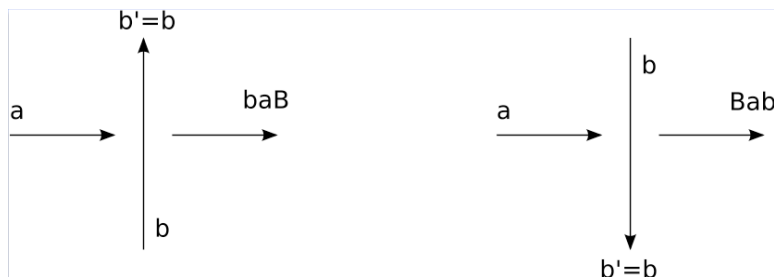


Figure 1: a, b, b' denote meridians of arcs at a ‘negative’ crossing (left) and a ‘positive’ crossing (right). Loops around arcs are oriented using the right-hand rule. Capitals denote inverses: $B := b^{-1}$.

The standard Wirtinger presentation of the fundamental group of a link complement in the 3-sphere S^3 arising from a planar link projection with N crossings involves N generators and N relations. Generators correspond to small meridian loops linking each of the N oriented over-arcs of a projection, oriented according to the right-hand rule; the corresponding 4-valent planar graph has N vertices and $2N$ arcs, which combine to create the N overarcs.

- Begin with $n + k$ concentric circles in the plane, labeled $g_1, \dots, g_n, r_1, \dots, r_k$ as indicated in Figure 2, with anticlockwise orientation. These are in correspondence with the n generators and k relations of P_G , with g_i corresponding to X_i , r_j to R_j . We refer to components of the link accordingly as *generator components* and *relation components*.
- Concentrically divide each circle into L equal length arcs. To each of the L segments there corresponds a unique monomial $X_i^{\pm 1}$ occurring in a unique relation R_j of P_G :

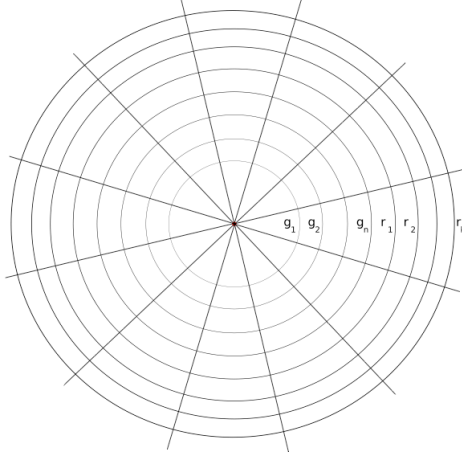


Figure 2: $n + k = 8$ labelled concentric circles, each decomposed into $L = 12$ arcs by rays from the origin.

as we read the monomials $X_i^{\pm 1}$ in the concatenated word $R_1 \cdots R_j \cdots R_k$ from left to right, we move anticlockwise from one segment to the next.

- Each relation R_j corresponds to l_j consecutive segments of r_j . We will make L modifications of the link, using the same procedure in each segment: if X_i^ϵ is associated to a given segment of r_j , we create simple linking between the components arcs of g_i and r_j within the segment, with linking number $\epsilon = \pm 1$. This is illustrated in Figure 3, where we have conveniently substituted horizontal line segments for the concentric circular arc segments, labeled $g_1, \dots, g_n, r_1, \dots, r_k$ read top to bottom. Note that we maintain the labels assigned to components of the original unlink, and here add labels to overarcs of link components corresponding to their respective meridians in the fundamental group of the link complement.
- Having made L modifications, we obtain a link $\mathcal{L}_{P_G} = \mathcal{G}_{P_G} \cup \mathcal{R}_{P_G}$ of $n+k$ components, and N crossings, which is naturally the union of two tangled unlinks \mathcal{G}_{P_G} \mathcal{R}_{P_G} , respectively the generator and relation components. The link \mathcal{L}_{P_G} has the following properties:
 - (a) Components remain planar, unknotted, and parametrized by the standard angles of the unit circle, giving a closed $(n+k)$ -stranded pure braid;
 - (b) Finitely many crossings occur, with at most one for any given angle;

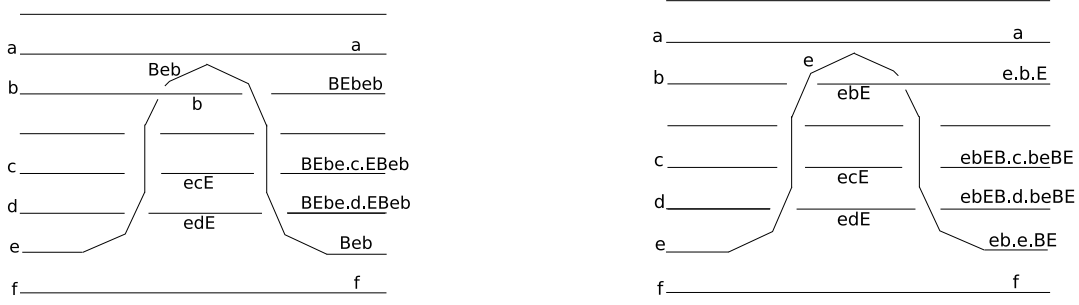


Figure 3: Local linking from the monomials X_i^{+1} (left) and X_i^{-1} (right) in the relation R_j . The left arc of generator component g_i has meridian label $b \in \pi_1(S^3 - \mathcal{L}_{P_G})$; the relation component r_j has left arc labeled $e \in \pi_1(S^3 - \mathcal{L}_{P_G})$. Components are oriented left to right: a, b, c correspond to meridians of generator component arcs; d, e, f are meridians of relation component arcs. Labels of arcs on the right of each figure arise from the convention for Wirtinger relations.

(c) The set of plane-projected crossings decompose all component circles g_i, r_j into subarcs with corresponding meridians labeled either $g_{i,s}$ or $r_{j,t}$, where the second subscripts are consecutive along each oriented circle, beginning with the first-labeled subarc corresponding to the segment at standard angle $0 \equiv 2\pi$. The total number of subarcs is $2N$, and at each crossing, the two subarcs of the overarc have equal meridian generators of $\pi_1(S^3 - \mathcal{L}_{P_G})$ assigned to them. The Wirtinger relations determine how consecutive meridians assigned to underarcs are conjugate to each other;

(d) Crossings are ordered by their order of occurrence in the concatenation of relations, $R_1 \cdots R_j \cdots R_k$.

- Generator and relation components may cross and link each other; two relation components may cross but remain unlinked; but generator components do not cross each other. Collectively both the set of generator components, and the set of relation components, continue to form unlinks of respectively n and k components, with complements in S^3 having free fundamental group respectively of ranks n and k . There are thus six types of crossings of interest in the link projection, according to the sign of the crossing, and which of a generator or relation component creates the overarc.
- It is convenient to use a mild variation of the standard Wirtinger presentation, using $2N$ generators for symmetry purposes, and $2N$ relations, N of which being equivalent to the standard relations, and the additional N asserting equalities for the additional generators.
- For any given component of any oriented link in S^3 , corresponding generators for the

Table 1: Wirtinger meridian data for six crossing types involving oriented components g_i, r_j, r_m

$g_{i,s}$ over $r_{j,t}$	$g_{i,s}$ over $r_{j,t}$	$r_{j,t}$ over $g_{i,s}$	$r_{j,t}$ over $g_{i,s}$	$r_{j,t}$ over $r_{m,u}$	$r_{j,t}$ over $r_{m,u}$
$+: A_{i,s;j,t}$	$-: B_{i,s;j,t}$	$+: C_{j,t;i,s}$	$-: D_{j,t;i,s}$	$+: E_{j,t;m,u}$	$-: F_{j,t;m,u}$
$g_{i,s+1} = g_{i,s}$	$g_{i,s+1} = g_{i,s}$	$r_{j,t+1} = r_{j,t}$	$r_{j,t+1} = r_{j,t}$	$r_{j,t+1} = r_{j,t}$	$r_{j,t+1} = r_{j,t}$
$r_{j,t+1} = g_{i,s}^{-1} r_{j,t} g_{i,s}$	$r_{j,t+1} = g_{i,s} r_{j,t} g_{i,s}^{-1}$	$g_{i,s+1} = r_{j,t}^{-1} g_{i,s} r_{j,t}$	$g_{i,s+1} = r_{j,t} g_{i,s} r_{j,t}^{-1}$	$r_{m,u+1} = r_{j,t}^{-1} r_{m,u} r_{j,t}$	$r_{m,u+1} = r_{j,t} r_{m,u} r_{j,t}^{-1}$

(generalized) Wirtinger presentation are all conjugate to each other. Table 1 enables us to record how meridian labels for arcs for generator and relation components are explicitly conjugate to each other, and thus to record contributions to words expressing longitudes – planar parallels for any given component – as products of conjugates of meridians. By potential abuse of notation, the labelling of arcs by meridians $g_{i,j}, r_{j,t}$ reflects the naming of components g_i, r_j .

Assembling the data from each crossing, we obtain:

Lemma 3. *The defining Wirtinger presentation of $\pi_1(S^3 - \mathcal{L}_{P_G})$ can be expressed*

$$\pi_1(S^3 - \mathcal{L}_{P_G}) := \langle g_{i,s}, r_{j,t} \mid A_{i,s;j,t}, B_{i,s;j,t}, C_{j,t;i,s}, D_{j,t;i,s}, E_{j,t;m,u}, F_{j,t;m,u} \rangle$$

where each symbol $X_{a,b;c,d}$ denotes the appropriate pair of relations for a crossing read off from Table 1.

3 Proof of Theorem 1

To complete the proof, we identify longitudes for each component, and a distinguished meridian, corresponding to a choice of generators for the fundamental groups of peripheral tori. The use of Tietze transformations allows us to identify the quotient groups obtained by adding relations trivializing appropriate meridians and longitudes, to obtain the initially given presentation P_G . Accordingly we add generators to the presentation corresponding

to a choice of meridian and longitude for each peripheral torus, and relations expressing these in terms of existing meridian generators for the link complement.

Meridian generators: To each subset of generators $\{g_{i,s}\}_{s=1}^{s_i}$, there corresponds a new generator X_i and defining relation $X_i := g_{i,1}$; similarly, to $\{r_{j,t}\}_{t=1}^{t_j}$ there corresponds a new generator \bar{R}_j and defining relation $\bar{R}_j := r_{j,1}$. The Wirtinger presentation of $\pi_1(S^3 - \mathcal{L}_{P_G})$ can be modified by Tietze transformations to $\pi_1(S^3 - \mathcal{L}_{P_G}) :=$

$$\langle g_{i,s}, r_{j,t}, X_i, \bar{R}_j \mid A_{i,s;j,t}, B_{i,s;j,t}, C_{j,t;i,s}, D_{j,t;i,s}, E_{j,t;m,u}, F_{j,t;m,u}, X_i g_{i,1}^{-1}, \bar{R}_j r_{j,1}^{-1} \rangle$$

Longitudes generators: Each component is planar, unknotted, and hence admits a planar-parallel longitude in $S^3 - \mathcal{L}_{P_G}$. Components are of two kinds – corresponding to generators and relations of the given presentation – which are to be respectively surgered using the framing of a longitude, or coned so that meridians and longitudes become trivial. The natural construction of the link leads to a simple product description for longitudes, corresponding to the product structure of concatenated relations given by the presentation. A longitude for a component g_i will be denoted γ_i , and for a component r_j , we denote a longitude by ρ_j .

To express γ_i, ρ_j as words in $\{g_{i,s}, r_{j,t}\}$, we follow the longitudes around the link, consecutively recording each oriented meridian generator corresponding to overarcs where the longitude passes under a component of \mathcal{L}_{P_G} . Each longitude is thus a product of L subwords, corresponding to the occurrences of original generators $X_i^{\pm 1}$ in the relation-concatenation $R_1 \cdots R_j \cdots R_k$. Using the Wirtinger relations, as in Figure 3, we see that the involved arc of the longitudes parallel to components labeled a, b, c, d, e, f contributes subwords given by Table 2 and Table 3, with capitals again denoting inverses.

Table 2: Wirtinger longitude subword contributions from the occurrence of X_i in R_j

Component	a	b	c	d	e	f
Contribution	1	Beb	E.Beb	E.Beb	b	1
If $d, e, f = 1$	1	1	1	1	$b = X_i$	1

Let $M_{\mathcal{L}_{P_G}}$ denote the compact exterior of the disjoint union of open solid donut neighbourhoods of all link components of \mathcal{L}_{P_G} , and let $M_{C\mathcal{R}_{P_G}}$ denote the link-singular 3-

Table 3: Wirtinger longitude subword contributions from the occurrence of X_i^{-1} in R_j

Component	a	b	c	d	e	f
Contribution	1	E	E.ebeBE	E.ebeBE	EBe	1
If $d, e, f = 1$	1	1	1	1	$B = X_i^{-1}$	1

manifold obtained by attaching a cone to each boundary torus of relation components. Thus all meridian generators $r_{j,t}$ and longitudes ρ_j of components r_j are set equal to 1 as relations added to the presentation of $\pi_1(S^3 - \mathcal{L}_{P_G})$, by the Seifert-van Kampen Theorem.

Lemma 4. *The group $\pi_1(M_{C\mathcal{R}_{P_G}})$ admits a finite presentation $:= \langle X_i | \rho_j \rangle$.*

Proof: We apply Tietze transformations to the presentation of $\pi_1(M_{C\mathcal{R}_{P_G}}) :=$

$$\begin{aligned} &\langle g_{i,s}, r_{j,t}, X_i, \bar{R}_j | A_{i,s;j,t}, B_{i,s;j,t}, C_{j,t;i,s}, D_{j,t;i,s}, E_{j,t;m,u}, F_{j,t;m,u}, X_i g_{i,1}^{-1}, \bar{R}_j r_{j,1}^{-1}, r_{j,t}, \rho_j \rangle \\ &:= \langle g_{i,s}, X_i, \bar{R}_j | g_{i,s+1} = g_{i,s}, X_i g_{i,1}^{-1}, \bar{R}_j, \rho_j \rangle := \langle X_i, \bar{R}_j | \bar{R}_j, \rho_j \rangle := \langle X_i | \rho_j \rangle. \end{aligned}$$

We now identify the words defining each ρ_j in $\pi_1(M_{C\mathcal{R}_{P_G}})$, using the tables. From these, we have:

Lemma 5. (a) *The group $\pi_1(M_{C\mathcal{R}_{P_G}})$ admits a finite presentation $:= \langle X_i | R_j \rangle$. (b) In $\pi_1(M_{C\mathcal{R}_{P_G}})$, $\gamma_i := 1$.*

Now perform 0-surgery on each generator component of \mathcal{L}_{P_G} , to obtain a closed link-singular 3-manifold M_{P_G} . This is achieved by attaching a solid donut $S^1 \times D_i^2$ to each of the remaining boundary tori of $M_{C\mathcal{R}_{P_G}}$, with the boundary of a D_i attached with framing determined by $\gamma_i = 1 \in \pi_1(M_{C\mathcal{R}_{P_G}})$. These elements are already trivial, and we obtain:

Theorem 3. *The fundamental group $\pi_1(M_{P_G})$ has presentation $\pi_1(M_{P_G}) := \langle X_i | R_j \rangle$.*

Corollary 1. *Every finitely presentable group admits a representation as the fundamental group of a closed, orientable link-singular 3-manifold, obtained by adding cones to the toral boundary components of a link exterior in some closed orientable 3-manifold.*

Proof: Reverse the order of attaching cones and solid donuts to $S^3 - \mathcal{L}_{P_G}$. Cones are then attached to boundary tori of a link exterior in the connect-sum $\#_n S^1 \times S^2$, the result of 0-surgery on the unlink \mathcal{G}_{P_G} of n components.

4 Genus: characterising compact orientable 3-manifold groups

In [3], it is shown by a different construction that every finitely presentable group arises as the fundamental group of a ‘singular 3-manifold’, in infinitely many ways: this is constructed from an orientable compact 3-manifold by coning one boundary component to a point, producing the only non-manifold (singular) point. Note that the resulting singular 3-manifold may have additional boundary components.

Definition. (Aitchison and Reeves [3]) The *genus* of G , denoted $\text{genus}(G)$, is defined as the least possible genus of such a boundary component for any such singular 3-manifold realizing the group G .

Remark. A question raised in [3] was whether or not every group could be realized without additional boundary components. We have answered this in the affirmative above.

Definition. The *closed link genus*, denoted $\text{clg}(G)$, of a finitely presentable group G is the minimal number of boundary tori among all $M \in \mathcal{M}_0^3$ such that $G \cong \pi_1(M_{C\partial})$.

Remark. By Theorem 1, this is well defined for any finitely presentable group G .

Theorem 4. *A finitely presentable group G has $\text{clg}(G) = 0$ if and only if G is the fundamental group of a closed orientable 3-manifold.*

Proof: If $G \cong \pi_1(M)$ for some closed orientable 3-manifold M , then $\text{clg}(G) = 0$ follows from the definition. If $\text{clg}(G) = 0$, then G is realized by a closed link-singular 3-manifold M with no singular points, which is thus a closed orientable 3-manifold.

If G is the fundamental group of a compact orientable 3-manifold M with a boundary component of positive genus, then G is also the fundamental group of some closed link-singular 3-manifold $M_{C\partial}^*$, $M^* \in \mathcal{M}_0^3$, by the main theorem, generally not free of singularities (since otherwise G is also a closed 3-manifold group). Conversely, by allowing additional boundary components, it is conceivable that we may decrease the number of singular points required to realize G .

Theorem 5. *Suppose G admits a finite presentation with k relations. Then*

$$\text{genus}(G) \leq \text{clg}(G) \leq k.$$

Proof: By the construction above, we know there exists a closed link-singular 3-manifold with at most k singular points which realizes G . Thus $\text{clg}(G) \leq k$. In [3], it is shown that if there exists a singular 3-manifold M realizing G , with two singular points respectively arising from cones on surfaces of genus g_1, g_2 , then there is a singular 3-manifold M' realizing G with a single cone on a surface of genus $g_1 + g_2$ replacing these. Thus if there are k cones on tori in a closed link-singular 3-manifold M realizing G , there is a singular

3-manifold M' realizing G with exactly 1 singular point, arising from a surface of genus k . Thus $\text{genus}(G) \leq \text{clg}(G)$.

Example. Baumslag–Solitar groups with presentations of form $G := \langle x_1, x_2 \mid x_1 x_2^m x_1^{-1} x_2^{-n} \rangle$ were shown in [3] to have $\text{genus}(G) = 1$ when $|n|, |m| \neq 1$. Since they are 1-relator groups, we also have $\text{clg}(G) = 1$. It is well known that such groups are not fundamental groups of compact orientable 3-manifolds, with or without boundary [30].

5 Hyperbolic link exteriors suffice

Every non-compact hyperbolic 3-manifold M of finite volume is the interior of a compact 3-manifold $M \in \mathcal{M}_0^3$ with nonempty boundary a disjoint union of tori. We show that every finitely presentable group G can be realized by coning boundary components of the exterior of a link whose complement admitting a hyperbolic structure.

Theorem 6. *Suppose G is an arbitrary finitely presentable group. Then there are infinitely many $M \in \mathcal{M}_0^3$ with interior admitting a (unique) hyperbolic structure, such that $G \cong \pi_1(M_{C\partial})$.*

Proof: We recall the following theorem of Myers, quoted from [25]:

Theorem 7. (Myers) *Let M be a compact, connected 3-manifold. Suppose J is a compact (but not necessarily connected), properly embedded 1-manifold in M . J is homotopic rel ∂J to an excellent 1-manifold K if and only if J meets every 2-sphere in ∂M in at least two points and every projective plane in ∂M in at least one point. In this case there are infinitely many such K with nonhomeomorphic exteriors. Moreover, each K can be chosen so that it is ribbon concordant to J .*

An excellent 1-manifold is one whose complement is an excellent 3-manifold: Excellent 3-manifolds admit hyperbolic structures, i.e., Riemannian metrics on their interiors having constant sectional curvature -1. Heuristically, homotoping J to K punctures all homotopically essential spheres and tori without creating new ones (other than peripheral tori) in its complement.

To conclude the proof, consider $\mathcal{L}_{P_G} = \mathcal{G}_{P_G} \cup \mathcal{R}_{P_G}$. Carry out 0-surgery on the sublink of generating components \mathcal{G}_{P_G} , so that \mathcal{R}_{P_G} becomes a new link $J := \mathcal{R}'$ of relation components in $M = \#_n S^2 \times S^1$. Let $K = \mathcal{R}''$ be any excellent 1-manifold obtained from J by Myers' Theorem. Since \mathcal{R}'' is homotopic to \mathcal{R}' , after coning we find $\pi_1(M_{C\mathcal{R}'}) \cong \pi_1(M_{C\mathcal{R}''})$: this follows since meridians of relation components are set to 1, and so Wirtinger generators and relations corresponding to crossings of relation components are irrelevant to the presentation of these groups. We may change such crossings at will, corresponding to homotopy, without changing the resulting groups.

For G^* a Kleinian group of isometries of hyperbolic 3-space \mathbb{H}^3 , with $M_{G^*} \cong \mathbb{H}^3/G^*$ a non-compact N -cusped orientable 3-manifold of finite volume, let $\mathcal{P}_{G^*} \subset S_\infty^2 = \partial\mathbb{H}^3$ be its dense set of parabolic fixed points. Let $\bar{M}_{G^*} := \mathbb{H}^3 \cup \mathcal{P}_{G^*}/G^*$ be the 3-complex obtained by compactifying each cusp of M_{G^*} with an additional point. This is the 3-dimensional analogue of the standard compactification of cusps of hyperbolic Riemann surfaces. As corollaries to the last theorem, we have:

Theorem 8. *Every finitely presentable group G is of the form $G = \pi_1(\bar{M}_{G^*})$, in infinitely many ways.*

Corollary 2. *Any invariant of non-compact hyperbolic 3-manifolds of finite volume, whose values can be ordered, defines an invariant of finitely presentable groups, by minimizing values over all hyperbolic link complements for which the addition of cones to their exterior realizes any given group G .*

As an example:

Definition. The *volume* $\text{vol}(G)$ of a finitely presentable group G is the least volume of any hyperbolic link complement $\mathcal{L} \subset M$ such that $M_{\mathcal{L}}$ realizes G .

6 Concluding remarks

1. *Volume:* The trivial group arises by coning the boundary tori of the exterior of any link with hyperbolic complement in S^3 , since the fundamental group of the complement is generated by meridians. When G is the trivial group, $\text{vol}(G)$ is of course bounded by the smallest volume of any hyperbolic link complement in S^3 . What can be said of the groups realized by coning on the smallest volume hyperbolic link complements in $\#_n S^2 \times S^1$?
2. *Relative hyperbolicity:* Gromov's definition of relatively hyperbolic groups is motivated by the fundamental groups and peripheral subgroups of hyperbolic link complements. The construction of the previous section shows the naturality of this concept and the direct use of the hyperbolic geometry of link complements to understand finitely presentable groups, arising by Dehn surgery on link complements – see for example Lackenby, and Fujiwara and Manning [19, 6].
3. *Energy concepts:* A simple ‘generalization’ of \mathcal{L}_{P_G} is a tangle $\mathcal{L} = \mathcal{G} \cup \mathcal{R}$ of two unlinks with respectively n and k components. Perform 0-surgery on components of \mathcal{G} , and cone on components of \mathcal{R} . The resulting fundamental group can be read off from the linking data of each component of \mathcal{R} with components of \mathcal{G} , and so this construction is essentially equivalent to a homotopy of the components of \mathcal{R}_{P_G} in the complement of \mathcal{G}_{P_G} . It would be interesting to use some notions of knot/link energies to find some canonical, or a finite number of possible links, representing a

given group G by minimizing some ‘relative’ link-homotopy energy associated to the linking of the unlinks \mathcal{G}, \mathcal{R} , derived from such as the original one first introduced by Fukuhara [7]. Given a presentation for a group G , there should be only finitely many possibilities, and generically perhaps one. When is such a link hyperbolic, and in such cases, what is its volume compared to $\text{vol}(G)$?

4. *Braids*: Our construction of \mathcal{L}_{P_G} produces a link which is a pure braid: for example, the Baumslag–Solitar groups with presentations of form $G := \langle x_1, x_2 \mid x_1 x_2^m x_1^{-1} x_2^{-n} \rangle$ give 3-braids which are of very simple form: Recall that link is called *quasipositive* [29] if it is the closure of a braid which is the product of conjugates of the Artin generators $\sigma_i^{\pm 1}$. This concept has its origins in links of singularities of algebraic curves in $\mathbb{C}^2 \cong \mathbb{R}^4$, locally related to cones on knots and links. It would be interesting to relate these concepts with \mathcal{L}_{P_G} for a given P_G , which is the product of conjugates of squares $\sigma_i^{\pm 2}$, when exponents are positive: such is the case when $m, -n > 1$ above. For $n, m > 1$, the braids are alternating, and are thus fibered links with hyperbolic complements. Is it possible to find tangled unlink representatives for all finitely presentable groups, which are pure braids, so that their complements, before coning or surgery, are fibered and hyperbolic? In another direction, braid groups themselves are finitely presentable fundamental groups of hyperplane complements, and have been recently considered in the context of cryptography. Further comments on decision problems are given below.
5. *4-dimensional considerations*: Invariants of finitely presentable groups have naturally been considered in the context of invariants of 4-manifolds: see for example Kotschick [17, 18]. As an example of an invariant for a finitely-presentable group G , the Hausmann–Weinberger invariant [8] is defined as the minimal Euler characteristic $q(G)$ of a closed orientable 4-manifold M with fundamental group G : for a given M , the Euler characteristic $\chi(M)$ is easily computed from a link description of M , but generally such calculations only give upper bounds for such invariants. Our invariants can be considered as a lower-dimensional ‘analogue’ of this. In this note we proved all results within the realm of classical 3-dimensional topology: a more natural setting for some is in the theory of 4-manifolds, which will be discussed in a subsequent note.
6. *‘Geography’*: For a fixed finitely presentable group G , characterize the set of hyperbolic link complements corresponding to G .
7. *Recent quantum and other invariants*: The results of this paper lead naturally to the reinterpretation of recent 3-manifold invariants in the context of combinatorial and geometric group theory, and in ‘virtual’ knots and links.
8. *Number theory*: We see that our construction makes contact with classical constructions from Riemann surface theory and the number theory of hyperbolic geome-

try. For hyperbolic Riemann surfaces, compactification of cusps is natural, and relates number theory with parabolic fixed points added to the circle at infinity of the hyperbolic plane: this construction is fundamental, pertaining to ‘Monstrous Moonshine’, and the Taniyama–Shimura–Weil Conjecture. Characterize finitely presentable groups arising from *arithmetic* link complements – see Maclachlan and Reid [23], and Neumann and Reid [26].

9. *Decision problems:* The following well known theorems raise the problem of identifying certain hyperbolic link complements which are potentially very interesting. Find ‘nice’ representative link complements for finitely presented groups having the properties of the following theorems: For a survey of decision problems in group theory, see [2, 22].

Theorem 9. (Higman [10]) *There is a finitely presented group F that is universal for all finitely presented groups. This means that for any finitely presented group G there is a subgroup of F isomorphic to G .*

There is an analogue – *universal links* – in 3-manifold topology: the arithmetic Borromean link $\mathcal{B} \subset S^3$ is universal [11], as the set of its branched-covers includes all closed orientable 3-manifolds. Suppose $F \cong \pi_1(M_{C\partial}^F)$, $M^F \in \mathcal{M}_0^3$, and G is a subgroup of F . Then there is a covering $\pi : M_{C\partial}^G \rightarrow M_{C\partial}^F$, with $G \cong \pi_1(M_{C\partial}^G)$, inducing a covering $\pi : M^G := \pi^{-1}(M^F) \rightarrow M^F$. A torus boundary component of M^F may be multiply covered, but restricting to any connected component of its preimage in M^G , the covering is a homeomorphism: such a 3-manifold M^F has a very rich subclass of covering spaces of this form.

In the ‘opposite direction’, we recall a theorem proved by C. F. Miller III:

Theorem 10. (Miller [24]) *There is a finitely presented group all of whose non-trivial quotient groups have insoluble word problem.*

Reconcile decision algorithms based on Tietze transformations of presentations with the recent proof of Thurston’s Geometrization Conjecture: note trisection of an angle *is* possible using ruler and compass constructions, *allowing an additional marked point*: It is perhaps fruitful to now construe finitely presentable groups as fundamental groups of link-singular manifolds, and define them as such rather than via presentations.

10. *Presentation link calculus:* For surgery representations of 3-manifolds, stabilization and handle-sliding generate the equivalence relation between framed links yielding the same 3-manifold. This creates an analogue of Tietze moves generating the equivalence relation between presentations of the same group: In the coned-link representation of finitely presentable groups, we can handle slide generating components over each

other, and similarly relation components over each other. Relation components can slide over generating components, but not vice-versa. We can stabilize by adding a Hopf link separated from other components, consisting of a generator and a relation component, or add a new relation component, unknotted and separated from the original link. Relation components can be homotoped arbitrarily in the complement of generating components, which is equivalent to homotopy in a connect-sum $\#_n S^2 \times S^1$ obtained by 0-surgery on generator components: observe that (a) the complement in S^3 of an n -component unlink has fundamental group which is free of rank n , as is $\pi_1(\#_n S^2 \times S^1)$; and (b) sliding a relation component over a 0-surgered generator component yields a new link obtainable by homotopy of the relation component in the complement of the generator components, since all generator components are unknotted and unlinked.

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